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Nth-degree spline Method for Solving Dirichlet Condition (DC) of Linear Ordinary Differential Equations (ODEs)

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ASTRACT

The aim of this paper is to find approximate solution of linear ordinary differential equations (ODEs) with dirichlet condition (DC) by using cubic B-spline method.

الخلاصة

الهدف من البحث هو ايجاد الحلول التقريبية للمعادلات التفاضلية الخطية مع شروط ديراشليت بأستخدام طريقة ب-شبه الخطي.

INDEX: Cubic B-spline function, Dirichlet Condition.

1. Introduction

A spline are the most popular. They produce in interpolated function that is continuous through to the second derivative. B-splines are the basis functions that satisfy our continuity conditions.

Splines tend to be stapler than fitting a polynomial through the (N+1) points, with less possibility of wild oscillations between the tabulated points. [1],[4]

In the present paper, a cubic b-spline is used to solve two point dirichlet conditions as following linear systems which are assumed to have a unique solution in the interval [0,1]

$$y''(x) + m(x)\, y'(x) + n(x)\, y(x) = f(x) \qquad o \le x \le 1 \qquad (1)$$

With dirichlet conditions

y (0) =0 y (1) =0

Where m(x), n(x), and f(x) are continuous function, we suppose that n(x)=m(x)=1

In part (2), we have given the definition of Nth-degree spline, this method presents to approximate the solution of two point dirichlet conditions.

In part (3) ,(4) and (5) we have solved problem using the method with two conditions

In the last part, report the major conclusion and further developments.

2. The Cubic B-spline

Let $\sigma = \{x_0, x_1, \dots, x_n\}$ be a set of partition of [0,1], the zero degree B-spline is defined as follow:[2]

$$B_{i,0}(x) = \begin{cases} 1 & x \in [x_1, x_{i+1}) \\ 0 & o.w \end{cases}$$

And for positive p, it is defined in the following recursive form:

$$B_{i,p}(x) = \frac{x - x_i}{x_{i+p} - x_i} B_{i,p-1}(x) + \frac{x_{i+p+1} - x}{x_{i+p+1} - x_{i+1}} B_{i+1,p-1}(x) \qquad p \ge 2$$

From this above we get the cubic B-spline:

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$$B_{0,3}(x)=\frac{1}{6h^3}\begin{cases} x^3 & x\in[0,h) \\ -3x^3+12hx^2-12h^2x+4h^3 & x\in[h,2h) \\ 3x^3-24hx^2+60h^2x-44h^3 & x\in[2h,3h) \\ -x^3+12hx^2-48h^2x+64h^3 & x\in[3h,4h) \end{cases}$$

Thus, the function s(x) is represent in the form

$$s(x)=\sum_{i=-3}^{n-1} C_i B_{i,3}(x)$$

3. Some Properties of B-spline Functions

The following properties are presented in details which needed later. [3]

a- Translation invariance:

$B_{i-1,p}(x)=B_{0,p}(x-(i-1)h) \qquad i=-3,-2,\dots.$

b- Compact supported

$B_{i,p}(x)=0 \qquad x\notin[x_i,x_{i+p+1})$

c- $B_{i,p}^{(k)}(x)=\frac{p!}{(p-k)!}\sum_{j=0}^{k}\alpha_{k-1,j}B_{i+j,p-k}$

4. Nth-degree spline

Suppose that the interval $[x_0, x_n]$ divided into n subinterval with knots $x_0, x_1, \dots, x_n$, the function u(x) in the interval above is represented by nth-degree spline in the form

$s(x)=a+b(x-x_0)+c(x-x_0)^2+\cdots+\sum_{i=-3}^{n-1}d_i(x-x_i)^N$

And

$$s'(x)=b+2c(x-x_0)+\cdots+N\sum_{i=-3}^{n-1}d_i\,(x-x_i)^{N-1}$$

$$s''(x)=2c+\cdots+N(N-1)\sum_{i=-3}^{n-1}d_i\,(x-x_i)^{N-2}$$

$\vdots$

$$s^{(n)}(x)=N!\sum_{i=-3}^{n-1}d_i\,(x-x_i)$$

Where N is the degree of spline and equal 0,1,2,…

If N=7

$$s(x)=a_i(x-x_i)^7+b_i(x-x_i)^6+c_i(x-x_i)^5+d_i(x-x_i)^4+e_i(x-x_i)^3+f_i(x-x_i)^2+g_i(x-x_i)+z_i$$

Where i=0,1,…,n-1 , and $x\in[x_i,x_{i+1}]$

The nth-degree spline $s(x)\in C^{(n)}[a,b]$

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Then

$$s'(x) = 7a_i(x-x_i)^6 + 6b_i(x-x_i)^5 + 5c_i(x-x_i)^4 + 4d_i(x-x_i)^3 + 3e_i(x-x_i)^2 + 2f_i(x-x_i) + g_i$$

$\vdots$

$s^{(7)}(x) = 5040a_i$, where a_i is coefficient in the matrix A

4. Approximate Function for Solving DC of Linear Ordinary Differential Equations(ODEs)

Let $y(x) = \sum_{i=-3}^{n-i} C_i B_{i,N}(x)$ (2)

Be an approximate solution of equation (1), where $C_i$is unknown real coefficient and $B_{i,N}(x)$ are Nth-degree spline functions.

Let $x_0, x_1, \dots, x_n$ are (n+1) grid points in the interval [0,1], so that

$x_j = a + jh \qquad j = 0,1,\dots,n$

And $x = x_j$, $x_0 = a = 0$, $x_n = b = 1$, $h = \frac{b-a}{n}$

It is required that the approximate solution (2) satisfy equation at

$x = x_j$

By equation (1), we get:

$$\sum_{i=-3}^{n-1} C_i \left[B''_{i,N}(x_j) + B'_{i,N}(x_j) + B_{i,N}(x_j)\right] = f(x_j) \qquad j = 0,1,\dots,n \quad (3)$$

And dirichlet condition can be written as

$\sum_{i=-3}^{n-1} C_i B_{i,N}(0) = 0 \qquad for\ \ x = 0$ (4)

$\sum_{i=-3}^{n-1} C_i B_{i,N}(1) = 0 \qquad for\ \ x = 1$ (5)

The spline solution of (1) is obtained by solving the following matrix equation.

Then a system of (n+3) linear equations in the (n+3) unknowns

C_{-3}, C_{-3},…, C_{n-1} are obtained,

Using (2) can obtain the numerical solution

This systems can be written in the matrix vector as follow:

AC=F (6)

Where

$C=[C_{-3}, C_{-3},\dots, C_{n-1}]^T$

$F=[0,f(x_0), f(x_1), \dots, f(x_n),0]^T$

And A is (n+3)(n+3) dimensional tri-diagonal matrix given by:

$$A = \begin{bmatrix} 1 & 4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_0(x_0) & b_0(x_0) & c_0(x_0) & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_1(x_1) & b_1(x_1) & c_1(x_1) & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & a_n(x_n) & b_n(x_n) & c_n(x_n) \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 4 & 1 \end{bmatrix} . \quad (7)$$

Also the coefficients in the matrix A, where m(x) =n(x) =1

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We get,

$a_j(x_j) = \frac{6}{h^2} - \frac{3}{h} + 1$, j=0,1,…,n

$b_j(x_j) = -\frac{12}{h^2} + 4$, j=0,1,…,n

$c_j(x_j) = \frac{6}{h^2} + \frac{3}{h} + 1$, j=0,1,…,n

Then, a system of linear equations can be build as shown below:

$$\begin{bmatrix} 1 & 4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_0(x_0) & b_0(x_0) & c_0(x_0) & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_1(x_1) & b_1(x_1) & c_1(x_1) & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & a_n(x_n) & b_n(x_n) & c_n(x_n) \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} c_{-3} \\ c_{-2} \\ \vdots \\ \vdots \\ c_{n-2} \\ c_{n-1} \end{bmatrix}$$

$$= 6\begin{bmatrix} 0 \\ f(x_0) \\ \vdots \\ \vdots \\ f(x_n) \\ 0 \end{bmatrix}$$

5. Numerical Example

In this part, we introduce the numerical example to solve DC of linear ordinary differential equations (ODEs) by using Nth-degree spline method.

Example: solve the following DC by Nth-degree spline method where N=3

$y''(x) - y'(x) = -e^{x-1} - 1, \quad 0 \le x \le 1$

With DC: y (0) =0, y (1) =0

The analytical solution:

$y(x) = x(1 - e^{x-1})$

We can get coefficient matrix A by using (7) for n=10

$$A = \begin{bmatrix} 1 & 4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ (6+3h)/h^2 & -12/h^2 & (6-3h)/h^2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & (6+3h)/h^2 & -12/h^2 & (6-3h)/h^2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & (6+3h)/h^2 & -12/h^2 & (6-3h)/h^2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 4 & 1 \end{bmatrix}$$

And

F= [0,-8.2073,-8.4394,…,-11.4290,-12.0000,0]^T

Then, if h=0.1, we can get the function

C=[-0.0657,0.0012,0.0608,…,0.0050,-0.1102]^T

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And by using eq. (2) we can find the numerical solution by the B-spline method ,they follow that:

Y_1=0.0593827 , Y_2=0.110234 , Y_3=0.1512 , Y_4=0.1806167
Y_5=0.1969833 , Y_6=0.19808 , Y_7=0.1816 , Y_8=0.1452 , Y_9=0.08571

The analytical solutions are given by

$y_1(x_1) = 0.0593$, $y_2(x_2) = 0.1101$, $y_3(x_3) = 0.1510$,
$y_4(x_4) = 0.1805$, $y_5(x_5) = 0.1967$, $y_6(x_6) = 0.1978$,
$y_7(x_7) = 0.1814$, $y_8(x_8) = 0.1450$, $y_9(x_9) = 0.0856$.

And we can see the error

$y_1(x_1) - Y_1$=-0.0000827 , $y_2(x_2) - Y_2$=-0.000134
$y_3(x_3) - Y_3$=-0.0002 , $y_4(x_4) - Y_4$=0.000117
$y_5(x_5) - Y_5$=-0.0002833 , $y_6(x_6) - Y_6$=0.00023334
$y_7(x_7) - Y_7$=-0.00025523 , $y_8(x_8) - Y_8$=-0.0002
$y_9(x_9) - Y_9$=-0.00011

Then the max-absolute error is given by

Ω=0.00025523

6. Conclusion

The method B-spline considered for the numerical solution of DC of linear ordinary differential equations(ODEs), from the example we can say that is the better method for solving (ODEs) , also a spline function may be used to obtain the solution at any point in the range.

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